

# VARIATIONAL FORMULAS OF HIGHER ORDER MEAN CURVATURES

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**ABSTRACT.** In this paper, we establish the first variational formula and its Euler-Lagrange equation for the total  $2p$ -th mean curvature functional  $\mathcal{M}_{2p}$  of a submanifold  $M^n$  in a general Riemannian manifold  $N^{n+m}$  for  $p = 0, 1, \dots, [\frac{n}{2}]$ . As an example, we prove that closed complex submanifolds in complex projective spaces are critical points of the functional  $\mathcal{M}_{2p}$ , called relatively  $2p$ -minimal submanifolds, for all  $p$ . At last, we discuss the relations between relatively  $2p$ -minimal submanifolds and austere submanifolds in real space forms, as well as a special variational problem.

## 1. INTRODUCTION

It is well known that critical points of the volume functional for isometric immersions are submanifolds with vanishing mean curvature vector field. For a hypersurface, the mean curvature vector field is just given by the mean value of the principal curvatures (up to a direction). The higher order mean curvatures of a hypersurface are then defined as the (normalized) higher order elementary symmetric polynomials of the principal curvatures, whose variational properties were studied by Reilly [17] in real space forms and by Li [13] in general Riemannian manifolds. Reilly [18] also introduced the notion of higher order mean curvatures of compact submanifolds in Euclidean spaces when studying the first eigenvalue of the Laplacian. Moreover, he derived the first variational formula of the integral of each even order mean curvature. Afterwards, two natural generalizations came into intensive studies.

One natural way to define the higher order mean curvatures of a submanifold  $M^n$  in a general Riemannian manifold  $N^{n+m}$  is by using the curvature operator  $R^M$  (or the curvature forms  $\Omega_{ij}^M$ ) of the submanifold  $M$ , in which case the  $2p$ -th mean curvature and  $(2p+1)$ -th mean curvature vector field will be denoted by  $K_{2p}^M, H_{2p+1}^M$ . The other way is to use the relative curvature operator  $R^M - R^N$  (or the relative curvature forms  $\Omega_{ij}^M - \Omega_{ij}^N$ ) of the immersion  $f$  and the corresponding higher order mean curvatures will be denoted by  $K_{2p}^f, H_{2p+1}^f$ . See section 2 for explicit definitions. Note that  $H_1^M = H_1^f$  is just the mean curvature vector field, for hypersurfaces  $K_{2p}^f, H_{2p+1}^f$  are just the usual higher order mean curvatures, and for submanifolds in Euclidean spaces  $K_{2p}^M = K_{2p}^f, H_{2p+1}^M = H_{2p+1}^f$  are just the higher order mean curvatures defined by Reilly. In general,  $K_{2p}^M$  depends only on the metric of the submanifold and thus is an intrinsic invariant. It is called the  $2p$ -th Gauss-Bonnet curvature by Labbi [12] and its integral is called a Killing invariant by Li [14]. Both of Li [14] and Labbi [12] studied the variational problem of these intrinsic invariants and characterized the critical points by the vanishing of  $H_{2p+1}^M$  which thereby naturally generalize minimal submanifolds in a general Riemannian manifold into  $2p$ -minimal. On the other hand,  $K_{2p}^f$  is not intrinsic in general. Nevertheless, for submanifolds in

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2000 *Mathematics Subject Classification.* 53C42, 53C40.

*Key words and phrases.*  $2p$ -minimal, mean curvature, austere submanifold.

The project is partially supported by the NSFC (No.11001016), the SRFDP, and the Program for Changjiang Scholars and Innovative Research Team in University.

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real space forms, it can be expressed as a linear combination of  $1, K_2^M, \dots, K_{2p}^M$  and hence is intrinsic in this case. Among other things, Li [13] calculated the first variational formula of the integral of  $K_{2p}^f$  for submanifolds in real space forms and for hypersurfaces in general Riemannian manifolds. In analogy, Cao and Li [2] considered the variational problem of the integral of some linear combination of  $K_{2p}^f$ s for submanifolds in real space forms so as to characterize the critical points by the vanishing of  $H_{2p+1}^f$ , which they also called  $2p$ -minimal submanifolds. In addition, they obtained a non-existence result for closed stable  $2p$ -minimal submanifolds in spheres that would reduce to a result of Simons [19] when  $p = 0$  (Some similar results for hypersurfaces have been recently obtained by [15]). In view of these two lines of developments, we come to consider the variational problem of the integral of  $K_{2p}^f$  for submanifolds in a general Riemannian manifold.

In this paper, we establish the first variational formula and its Euler-Lagrange equation for the functional  $\mathcal{M}_{2p}(f) := \int_M K_{2p}^f dV_M$  defined as the total  $2p$ -th mean curvature of a submanifold  $M^n$  in a general Riemannian manifold  $N^{n+m}$  for  $p = 0, 1, \dots, [\frac{n}{2}]$ . For hypersurface case this has been done by Li [13]. It is noteworthy to mention that the object in this variational problem is no longer an intrinsic invariant as in preceding references. As an example, we prove that closed complex submanifolds in complex projective spaces are critical for the functional  $\mathcal{M}_{2p}$  for all  $p$ , which we called relatively  $2p$ -minimal. At last, we discuss the relations between  $2p$ -minimal submanifolds and austere submanifolds in real space forms, as well as a special variational problem.

## 2. PRELIMINARIES

We begin with the definition of the  $2p$ -th mean curvature and  $(2p+1)$ -th mean curvature vector field. Throughout this paper, we adopt the notions used in [7].

Let  $M^n$  and  $N^{n+m}$  be Riemannian manifolds of dimension  $n$  and  $n+m$  respectively, and  $f : M^n \rightarrow N^{n+m}$  be an isometric immersion. Around each point in  $M$ , choose a local orthonormal frame  $\{e_1, \dots, e_{n+m}\}$  of  $TN$  such that  $\{e_1, \dots, e_n\}$  are tangent vectors of  $M$  while  $\{e_{n+1}, \dots, e_{n+m}\}$  are normal to  $M$ . Then we use  $\{\theta_A \mid 1 \leq A \leq n+m\}$  and  $\{\theta_{AB} \mid 1 \leq A, B \leq n+m\}$  to denote the corresponding dual 1-forms and connection 1-forms respectively. The following convention for indices will be used throughout this paper:

$$1 \leq i, j, k \leq n, \quad n+1 \leq \alpha, \beta, \gamma \leq n+m, \quad 1 \leq A, B, C \leq n+m.$$

The structure equations of  $N$  are given by

$$\begin{cases} d\theta_A = \sum_B \theta_{AB} \wedge \theta_B, & \theta_{AB} = -\theta_{BA}, \\ d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB} - \Omega_{AB}^N, \end{cases}$$

where the curvature forms  $\Omega_{AB}^N = \frac{1}{2} \sum_{C,D} R_{ABCD} \theta_C \wedge \theta_D$  and  $R_{ABAB}$  is the sectional curvature of  $N$  at the two plane  $e_A \wedge e_B$ . Comparing with the structure equations of  $M$

$$\begin{cases} d\theta_i = \sum_j \theta_{ij} \wedge \theta_j, & \theta_{ij} = -\theta_{ji}, \\ d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj} - \Omega_{ij}^M, \end{cases}$$

we define the relative curvature forms  $\Omega_{ij}$  of the immersion  $f$  by using Gauss equation

$$(2.1) \quad \Omega_{ij} := \Omega_{ij}^M - \Omega_{ij}^N = \sum_{\alpha} \theta_{i\alpha} \wedge \theta_{j\alpha}.$$

**Definition 2.1.** For  $p = 0, 1, \dots, [\frac{n}{2}]$ , the  $2p$ -th (relative) mean curvature  $K_{2p}^f$  and the  $(2p+1)$ -th (relative) mean curvature vector field  $H_{2p+1}^f$  of  $f$  are defined as follows (cf. [7]):

$$(2.2) \quad \begin{aligned} K_{2p}^f &= \frac{(n-2p)!}{n!} \sum_{I_{2p}} \Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{2p-1} i_{2p}} (e_{i_1}, \dots, e_{i_{2p}}), \\ H_{2p+1}^f &= \frac{(n-2p-1)!}{n!} \sum_{\alpha} \sum_{I_{2p+1}} \Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{2p-1} i_{2p}} \wedge \theta_{i_{2p+1} \alpha} (e_{i_1}, \dots, e_{i_{2p+1}}) e_{\alpha}, \end{aligned}$$

where the index  $I_k = (i_1, \dots, i_k)$  denotes  $k$  different integers in  $\{1, \dots, n\}$  for  $k = 1, \dots, n$ . We also denote  $K_0^f := 1$ ,  $H_{-1}^f := H_{n+1}^f := 0$ .

One can easily find that  $K_{2p}^f$  and  $H_{2p+1}^f$  are independent of the choice of the local frame and hence well-defined (cf. [7]). In analogy, the  $2p$ -th Gauss-Bonnet curvature  $K_{2p}^M$  and the  $(2p+1)$ -th mean curvature vector field  $H_{2p+1}^M$  introduced in last section can be defined by the same formulas of (2.2) with  $\Omega_{ij}^M$  instead of all  $\Omega_{ij}$  therein. When  $N^{n+m}$  is the real space form  $\mathbb{R}^{n+m}(c)$  of constant sectional curvature  $c$ , a straightforward calculation shows that the two families can express each other by

$$(2.3) \quad K_{2p}^M = \sum_{k=0}^p c^{p-k} \binom{p}{k} K_{2k}^f, \quad H_{2p+1}^M = \sum_{k=0}^p c^{p-k} \binom{p}{k} H_{2p+1}^f.$$

If  $M^n$  is compact, possibly with boundary, the total  $2p$ -th mean curvature of  $f$  is given by the integral

$$(2.4) \quad \mathcal{M}_{2p}(f) := \int_M K_{2p}^f dV_M.$$

We apply a variation of the immersion  $f$  as follows: Let  $I$  be the interval  $-\frac{1}{2} < t < \frac{1}{2}$ . Let  $F : M \times I \rightarrow N$  be a differentiable mapping such that its restriction to  $M \times t$  ( $t \in I$ ), is an immersion, denoted by  $f_t$ , and that  $F(x, 0) = f(x)$  for  $x \in M$ . Our aim is to evaluate the first variational formula of the functional  $\mathcal{M}_{2p}$  under such variations, that is to calculate

$$(2.5) \quad \left. \frac{d}{dt} \mathcal{M}_{2p}(f_t) \right|_{t=0}.$$

To treat with this type of variational problems, we would like to apply the moving frame method presented by Chern in [3]. Choose a local orthonormal frame field  $\{e_A(x, t)\}$  of  $TN$  over  $M \times I$  such that for every  $t \in I$ ,  $e_i(x, t)$  are tangent vectors to  $M_t := f_t(M) = F(M \times t)$  at  $(x, t)$  and hence  $e_{\alpha}(x, t)$  are normal vectors. Let  $\omega_A$ ,  $\omega_{AB}$  be the corresponding dual 1-forms and connection 1-forms of  $N$  over  $M \times I$ . Then they can be written as

$$(2.6) \quad \omega_i = \theta_i + a_i dt, \quad \omega_{\alpha} = a_{\alpha} dt, \quad \omega_{AB} = \theta_{AB} + a_{AB} dt,$$

where  $\theta_i, \theta_{AB}$  are linear differential forms in  $M$  with coefficients which may depend on  $t$ . For  $t = 0$  they reduce to the forms with the same notation on  $M$ . The vector  $\nu := \sum_A a_A e_A(x, 0) = \frac{d}{dt} F(x, t)|_{t=0}$  is called the deformation vector. We write the exterior differential operator  $d$  on  $M \times I$  as

$$d = d_M + dt \frac{\partial}{\partial t}.$$

Now by the definition of  $2p$ -th mean curvature, we have

$$(2.7) \quad K_{2p}^f dV_{M_t} = \frac{(n-2p)!}{n!} \sum_{I_{2p}} \Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{2p-1} i_{2p}} (e_{i_1}, \dots, e_{i_{2p}}) dV_{M_t}$$

$$\begin{aligned}
&= \frac{(n-2p)!}{n!} \sum_{I_{2p}} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}} (e_{i_1}, \dots, e_{i_{2p}}) \theta_1 \wedge \cdots \wedge \theta_n \\
&= \frac{1}{n!} \sum_{I_n} \delta_{I_n} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}} \wedge \theta_{i_{2p+1}} \wedge \cdots \wedge \theta_{i_n},
\end{aligned}$$

where  $\delta_{I_n} := \delta_{i_1, \dots, i_n}^{1, \dots, n}$  denotes the generalized Kronecker symbol. Similarly, we have

$$(2.8) \quad \langle H_{2p+1}^{f_t}, \nu \rangle dV_{M_t} = \frac{1}{n!} \sum_{\alpha} \sum_{I_n} a_{\alpha} \delta_{I_n} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}} \wedge \theta_{i_{2p+1} \alpha} \wedge \theta_{i_{2p+2}} \wedge \cdots \wedge \theta_{i_n}.$$

Define an  $n$ -form on  $M$

$$(2.9) \quad \Theta_{2p} = \sum_{I_n} \delta_{I_n} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}} \wedge \theta_{i_{2p+1}} \wedge \cdots \wedge \theta_{i_n}.$$

Then by (2.7) our variational problem (2.5) turns to

$$(2.10) \quad \left. \frac{d}{dt} \mathcal{M}_{2p}(f_t) \right|_{t=0} = \left. \frac{d}{dt} \int_{M_t} K_{2p}^{f_t} dV_{M_t} \right|_{t=0} = \frac{1}{n!} \int_M \frac{\partial}{\partial t} \Theta_{2p} \Big|_{t=0}.$$

### 3. VARIATIONAL FORMULA OF THE TOTAL $(2p)$ -TH MEAN CURVATURE

In this section we will calculate in detail the first variational formula of the total  $2p$ -th mean curvature  $\mathcal{M}_{2p}(f)$  in (2.4) by moving frame method.

From last section, it suffices to calculate formula (2.10). Recalling the definition of  $\Omega_{ij}$  in (2.1), we put  $\tilde{\Omega}_{ij} := \sum_{\alpha} \omega_{i\alpha} \wedge \omega_{j\alpha}$  where  $\omega_{i\alpha}$  is the connection 1-form given in (2.6). Then substituting  $\tilde{\Omega}_{ij}, \omega_i$  for  $\Omega_{ij}, \theta_i$  into (2.9) respectively, we can define an  $n$ -form  $\Psi_{2p}$  on  $M \times I$ :

$$(3.1) \quad \Psi_{2p} = \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n}.$$

It is easily seen from (2.6, 2.9) that

$$(3.2) \quad \Psi_{2p} = \Theta_{2p} + dt \wedge \Phi_{2p},$$

where

$$\begin{aligned}
(3.3) \quad \Phi_{2p} &= -2p \sum_{I_n, \alpha} \delta_{I_n} a_{i_{2p} \alpha} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-3} i_{2p-2}} \wedge \theta_{i_{2p-1} \alpha} \wedge \theta_{i_{2p+1}} \wedge \cdots \wedge \theta_{i_n} \\
&\quad + (n-2p) \sum_{I_n} \delta_{I_n} a_{i_{2p+1}} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}} \wedge \theta_{i_{2p+2}} \wedge \cdots \wedge \theta_{i_n}.
\end{aligned}$$

Then taking exterior differential of the equation (3.2) we get

$$(3.4) \quad d\Psi_{2p} = d_M \Theta_{2p} + dt \wedge \frac{\partial}{\partial t} \Theta_{2p} - dt \wedge d_M \Phi_{2p}.$$

On the other hand,  $d\Psi_{2p}$  can be calculated directly from (3.1) by using the structure equations of  $N$  as the following.

**Lemma 3.1.** *Notations as above, then*

$$\begin{aligned}
(3.5) \quad d\Psi_{2p} &= (n-2p) \sum_{\alpha} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge \omega_{i_{2p+1} \alpha} \wedge \omega_{\alpha} \wedge \omega_{i_{2p+2}} \wedge \cdots \wedge \omega_{i_n} \\
&\quad + 2p \sum_{\alpha} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \Omega_{i_{2p} \alpha}^N \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n}.
\end{aligned}$$

*Proof.* Using the structure equations of  $N$  and interchanging the indices whenever there occur two essentially equal terms, we can obtain the following expression:

$$\begin{aligned}
 d\Psi_{2p} &= -2p \sum_{\alpha} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge d\omega_{i_{2p} \alpha} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &\quad + (n-2p) \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge d\omega_{i_{2p+1}} \wedge \omega_{i_{2p+2}} \wedge \cdots \wedge \omega_{i_n} \\
 &= -2p \sum_{\alpha} \sum_{I_n, j} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \omega_{i_{2p} j} \wedge \omega_{j \alpha} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &\quad - 2p \sum_{\alpha, \beta} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \omega_{i_{2p} \beta} \wedge \omega_{\beta \alpha} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &\quad + 2p \sum_{\alpha} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \Omega_{i_{2p} \alpha}^N \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &\quad + (n-2p) \sum_{I_n, j} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge \omega_{i_{2p+1} j} \wedge \omega_j \wedge \omega_{i_{2p+2}} \wedge \cdots \wedge \omega_{i_n} \\
 &\quad + (n-2p) \sum_{\alpha} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge \omega_{i_{2p+1} \alpha} \wedge \omega_{\alpha} \wedge \omega_{i_{2p+2}} \wedge \cdots \wedge \omega_{i_n}.
 \end{aligned}$$

Since the index  $I_n = (i_1, \dots, i_n)$  is a permutation of  $\{1, \dots, n\}$ , the sum over  $j$  from 1 to  $n$  can be looked as from  $i_1$  to  $i_n$ , which leads to the following:

$$\begin{aligned}
 &-2p \sum_{\alpha} \sum_{I_n, j} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \omega_{i_{2p} j} \wedge \omega_{j \alpha} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &= 2p \sum_{I_n, j} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \tilde{\Omega}_{i_{2p-1} j} \wedge \omega_{i_{2p} j} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &= 2p(2p-2) \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \tilde{\Omega}_{i_{2p-1} i_{2p-2}} \wedge \omega_{i_{2p} i_{2p-2}} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &\quad + 2p(n-2p) \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \tilde{\Omega}_{i_{2p-1} i_{2p+1}} \wedge \omega_{i_{2p} i_{2p+1}} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &= 0 + 2p(n-2p) \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \tilde{\Omega}_{i_{2p-1} i_{2p+1}} \wedge \omega_{i_{2p} i_{2p+1}} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
 &= -2p(n-2p) \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge \omega_{i_{2p+1} i_{2p}} \wedge \omega_{i_{2p}} \wedge \omega_{i_{2p+2}} \wedge \cdots \wedge \omega_{i_n} \quad (i_{2p} \leftrightarrow i_{2p+1}),
 \end{aligned}$$

where the vanishing of the third line can be easily checked by exchanging the indices  $i_{2p-1}, i_{2p-3}$ . Similarly,

$$\begin{aligned}
 &(n-2p) \sum_{I_n, j} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge \omega_{i_{2p+1} j} \wedge \omega_j \wedge \omega_{i_{2p+2}} \wedge \cdots \wedge \omega_{i_n} \\
 &= 2p(n-2p) \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge \omega_{i_{2p+1} i_{2p}} \wedge \omega_{i_{2p}} \wedge \omega_{i_{2p+2}} \wedge \cdots \wedge \omega_{i_n}.
 \end{aligned}$$

Combining with

$$\sum_{\alpha, \beta} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \omega_{i_{2p} \beta} \wedge \omega_{\beta \alpha} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n}$$

$$\begin{aligned}
&= - \sum_{\alpha, \beta} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p} \beta} \wedge \omega_{i_{2p-1} \alpha} \wedge \omega_{\alpha \beta} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \quad \left( \begin{smallmatrix} i_{2p-1} \leftrightarrow i_{2p} \\ \alpha \leftrightarrow \beta \end{smallmatrix} \right) \\
&= - \sum_{\alpha, \beta} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \omega_{i_{2p} \beta} \wedge \omega_{\beta \alpha} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
&= 0,
\end{aligned}$$

we complete the proof of Lemma 3.1.  $\square$

From Lemma 3.1 we can divide the expansion of  $d\Psi_{2p}$  into two parts: one part involving  $dt$  and the other not. In what follows the part of  $dt$  in (3.5) will be calculated, since we want to get the expression of  $\frac{\partial}{\partial t} \Theta_{2p}$  concretely by comparing with the corresponding terms in (3.4).

Substituting into the first term of  $d\Psi_{2p}$  in (3.5) the expression of  $\omega_A, \omega_{AB}$  in (2.6) and recalling (2.8), we get

$$\begin{aligned}
(3.6) \quad & (n-2p) \sum_{\alpha} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-1} i_{2p}} \wedge \omega_{i_{2p+1} \alpha} \wedge \omega_{\alpha} \wedge \omega_{i_{2p+2}} \wedge \cdots \wedge \omega_{i_n} \\
&= -(n-2p) \sum_{\alpha} \sum_{I_n} a_{\alpha} dt \wedge \delta_{I_n} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}} \wedge \theta_{i_{2p+1} \alpha} \wedge \theta_{i_{2p+2}} \wedge \cdots \wedge \theta_{i_n} \\
&= -(n-2p)n! dt \wedge \langle H_{2p+1}^f, \nu \rangle dV_{M_t}.
\end{aligned}$$

Recall that  $\Omega_{i\alpha}^N = \frac{1}{2} \sum_{C,D} R_{i\alpha CD} \omega_C \wedge \omega_D$ . The second term of (3.5) turns to

$$\begin{aligned}
(3.7) \quad & 2p \sum_{\alpha} \sum_{I_n} \delta_{I_n} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \Omega_{i_{2p} \alpha}^N \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
&= 2p \sum_{\alpha, \beta} \sum_{I_n, j} \delta_{I_n} R_{i_{2p} \alpha j \beta} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \omega_j \wedge \omega_{\beta} \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
&\quad + p \sum_{\alpha} \sum_{I_n, j, k} \delta_{I_n} R_{i_{2p} \alpha j k} \tilde{\Omega}_{i_1 i_2} \wedge \cdots \wedge \tilde{\Omega}_{i_{2p-3} i_{2p-2}} \wedge \omega_{i_{2p-1} \alpha} \wedge \omega_j \wedge \omega_k \wedge \omega_{i_{2p+1}} \wedge \cdots \wedge \omega_{i_n} \\
&=: \Gamma_1 + \Gamma_2.
\end{aligned}$$

To simplify the notation, we put

$$(3.8) \quad \Omega_{I_{2p}} := \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}}.$$

Then we can get the expression of  $\Gamma_1$  as the following:

$$\begin{aligned}
\Gamma_1 &= 2p \sum_{\alpha, \beta} \sum_{I_n, j} \delta_{I_n} a_{\alpha} dt \wedge R_{i_{2p} \beta j \alpha} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1} \beta} \wedge \theta_j \wedge \theta_{i_{2p+1}} \wedge \cdots \wedge \theta_{i_n} \\
&= 2p(n-2p)! \sum_{\alpha, \beta} \sum_{I_{2p}, j} a_{\alpha} dt \wedge R_{i_{2p} \beta j \alpha} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1} \beta} \wedge \theta_j(e_{i_1}, \dots, e_{i_{2p}}) dV_{M_t} \\
&= \frac{2p(n-2p)!}{(2p-1)!} \sum_{\alpha, \beta} \sum_{I_{2p-1}, i} \sum_{J_{2p-1}, j} a_{\alpha} dt \wedge \delta_{j_1, \dots, j_{2p-1}, j}^{i_1, \dots, i_{2p-1}, i} R_{i \beta j \alpha} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1} \beta}(e_{j_1}, \dots, e_{j_{2p-1}}) dV_{M_t}.
\end{aligned}$$

Similarly, we can compute the  $dt$  part of  $\Gamma_2$  (denoted by  $\tilde{\Gamma}_2$ ) as well:

$$\begin{aligned}
\tilde{\Gamma}_2 &= -p(n-2p) \sum_{\alpha} \sum_{I_n, j, k} \delta_{I_n} a_{i_{2p+1}} dt \wedge R_{i_{2p} \alpha j k} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1} \alpha} \wedge \theta_j \wedge \theta_k \wedge \theta_{i_{2p+2}} \wedge \cdots \wedge \theta_{i_n} \\
&\quad - 2p \sum_{\alpha} \sum_{I_n, j, k} \delta_{I_n} a_j dt \wedge R_{i_{2p} \alpha j k} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1} \alpha} \wedge \theta_k \wedge \theta_{i_{2p+1}} \wedge \cdots \wedge \theta_{i_n}
\end{aligned}$$

$$\begin{aligned}
 & +p \sum_{\alpha} \sum_{I_n, j, k} \delta_{I_n} a_{i_{2p-1}\alpha} dt \wedge R_{i_{2p}\alpha j k} \Omega_{I_{2p-2}} \wedge \theta_j \wedge \theta_k \wedge \theta_{i_{2p+1}} \wedge \cdots \wedge \theta_{i_n} \\
 & -2p(p-1) \sum_{\alpha, \beta} \sum_{I_n, j, k} \delta_{I_n} a_{i_{2p-2}\beta} dt \wedge R_{i_{2p}\alpha j k} \Omega_{I_{2p-4}} \wedge \theta_{i_{2p-3}\beta} \wedge \theta_{i_{2p-1}\alpha} \wedge \theta_j \wedge \theta_k \wedge \theta_{i_{2p+1}} \wedge \cdots \wedge \theta_{i_n} \\
 = & \frac{p(n-2p)!}{(2p-1)!} \sum_{\alpha} \sum_{I_{2p-1}, i, i'} \sum_{J_{2p-1}, j, j'} a_i dt \wedge \delta_{j_1, \dots, j_{2p-1}, j, j'}^{i_1, \dots, i_{2p-1}, i, i'} R_{i'\alpha j j'} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\alpha} (e_{j_1}, \dots, e_{j_{2p-1}}) dV_{M_t} \\
 & - \frac{2p(n-2p)!}{(2p-1)!} \sum_{\alpha} \sum_{I_{2p-1}, i, i'} \sum_{J_{2p-1}, j, j'} a_i dt \wedge \delta_{j_1, \dots, j_{2p-1}, j, j'}^{i_1, \dots, i_{2p-1}, i, i'} R_{i'\alpha j j'} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\alpha} (e_{j_1}, \dots, e_{j_{2p-1}}) dV_{M_t} \\
 & + \frac{p(n-2p)!}{(2p-2)!} \sum_{\alpha} \sum_{I_{2p-2}, i, i'} \sum_{J_{2p-2}, j, j'} \left( a_{i\alpha} dt \wedge \delta_{j_1, \dots, j_{2p-2}, j, j'}^{i_1, \dots, i_{2p-2}, i, i'} R_{i'\alpha j j'} \Omega_{I_{2p-2}} (e_{j_1}, \dots, e_{j_{2p-2}}) dV_{M_t} \right. \\
 & \left. + 2(p-1) \sum_{\beta} a_{i\alpha} dt \wedge \delta_{j_1, \dots, j_{2p-2}, j, j'}^{i_1, \dots, i_{2p-2}, i, i'} R_{i'\beta j j'} \Omega_{I_{2p-4}} \wedge \theta_{i_{2p-3}\alpha} \wedge \theta_{i_{2p-2}\beta} (e_{j_1}, \dots, e_{j_{2p-2}}) dV_{M_t} \right).
 \end{aligned}$$

Now we are ready to give the first variational formula.

**Theorem 3.2.** *Let  $f : M^n \rightarrow N^{n+m}$  be an isometric immersion from a compact manifold  $M$  (possibly with boundary) into a Riemannian manifold  $N$ . Then for  $p = 0, 1, \dots, [\frac{n}{2}]$ , the first variational formula of the total  $2p$ -th mean curvature  $\mathcal{M}_{2p}(f)$  in (2.4) is given by*

$$\frac{d}{dt} \mathcal{M}_{2p}(f_t) \Big|_{t=0} = \int_M \left( \langle -(n-2p)H_{2p+1}^f + pW_{2p-1}, \nu \rangle + p \sum_i \langle Q_{2p-2}^i, \nabla_{e_i} \nu \rangle \right) dV_M + \frac{1}{n!} \int_{\partial M} \Phi_{2p}.$$

Here  $\nu$  is the deformation vector field,  $\nabla$  is the Levi-Civita connection of  $N$ ,  $H_{2p+1}^f$  is the  $(2p+1)$ -th mean curvature vector field,  $\Phi_{2p}$  is defined in (3.3),  $W_{2p-1}$  and  $Q_{2p-2}^i$  are defined by

$$\begin{aligned}
 W_{2p-1} &= \frac{(n-2p)!}{(2p-1)!n!} \sum_{\alpha, \beta} \sum_{I_{2p-1}, i} \sum_{J_{2p-1}, j} \delta_{j_1, \dots, j_{2p-1}, j}^{i_1, \dots, i_{2p-1}, i} R_{i\beta j \alpha} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\beta} (e_{j_1}, \dots, e_{j_{2p-1}}) e_{\alpha} \\
 &+ \frac{(n-2p)!}{(2p-1)!n!} \sum_{\alpha} \sum_{I_{2p-1}, i, i'} \sum_{J_{2p-1}, j, j'} \left( \sum_j \delta_{j_1, \dots, j_{2p-1}, j, j'}^{i_1, \dots, i_{2p-1}, i, i'} R_{i'\alpha j j'} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\alpha} (e_{j_1}, \dots, e_{j_{2p-1}}) \right. \\
 &\quad \left. - 2\delta_{j_1, \dots, j_{2p-1}, j, j'}^{i_1, \dots, i_{2p-1}, i, i'} R_{i'\alpha j j'} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\alpha} (e_{j_1}, \dots, e_{j_{2p-1}}) \right) e_i, \\
 Q_{2p-2}^i &= \frac{(n-2p)!}{(2p-2)!n!} \sum_{\alpha} \sum_{I_{2p-2}, i, i'} \sum_{J_{2p-2}, j, j'} \left( \delta_{j_1, \dots, j_{2p-2}, j, j'}^{i_1, \dots, i_{2p-2}, i, i'} R_{i'\alpha j j'} \Omega_{I_{2p-2}} (e_{j_1}, \dots, e_{j_{2p-2}}) \right. \\
 &\quad \left. + 2(p-1) \sum_{\beta} \delta_{j_1, \dots, j_{2p-2}, j, j'}^{i_1, \dots, i_{2p-2}, i, i'} R_{i'\beta j j'} \Omega_{I_{2p-4}} \wedge \theta_{i_{2p-3}\alpha} \wedge \theta_{i_{2p-2}\beta} (e_{j_1}, \dots, e_{j_{2p-2}}) \right) e_{\alpha},
 \end{aligned}$$

where  $\Omega_{I_{2p}} = \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}}$  is defined in (3.8), and  $W_{-1} = Q_{-2}^i = 0$ .

*Proof.* Comparing the parts involving  $dt$  of formulas (3.4) and (3.5) and substituting (3.6, 3.7) into (3.5), we obtain

$$(3.9) \quad \frac{1}{n!} \frac{\partial}{\partial t} \Theta_{2p} \Big|_{t=0} = \frac{1}{n!} d_M \Phi_{2p} + \langle -(n-2p)H_{2p+1}^f + pW_{2p-1}, \nu \rangle dV_M + p \sum_{i, \alpha} a_{i\alpha} q_{2p-2}^{i, \alpha} dV_M,$$

where  $q_{2p-2}^{i, \alpha}$  is the coefficient of  $e_{\alpha}$  in  $Q_{2p-2}^i$ , i.e.,  $Q_{2p-2}^i := \sum_{\alpha} q_{2p-2}^{i, \alpha} e_{\alpha}$ .

Recall that we have the following formula concerning the functions  $a_i, a_\alpha, a_{i\alpha}$  in (2.6) (cf. [3]):

$$\sum_i a_{i\alpha} \theta_i = d_M a_\alpha + \sum_\beta a_\beta \theta_{\beta\alpha} + \sum_i a_i \theta_{i\alpha} = D a_\alpha,$$

which implies immediately

$$\sum_{i,\alpha} a_{i\alpha} q_{2p-2}^{i,\alpha} = \sum_i \langle Q_{2p-2}^i, \nabla_{e_i} \nu \rangle.$$

Then taking use of (2.10) and integrating (3.9) over  $M$ , we complete the proof.  $\square$

**Remark 3.3.** Recalling the expression of  $\Phi_{2p}$  in (3.3), if we assume that  $M$  is closed or the variation satisfies  $a_i(x) = 0, a_{i\alpha}(x) = 0$  for  $x \in \partial M$ , the first variational formula turns to:

$$(3.10) \quad \left. \frac{d}{dt} \mathcal{M}_{2p}(f_t) \right|_{t=0} = \int_M \left( \langle -(n-2p)H_{2p+1}^f + pW_{2p-1}, \nu \rangle + p \sum_i \langle Q_{2p-2}^i, \nabla_{e_i} \nu \rangle \right) dV_M.$$

**Theorem 3.4.** Let  $f : M^n \rightarrow N^{n+m}$  be an isometric immersion from a closed Riemannian manifold  $M$  into a Riemannian manifold  $N$ . Then for  $p = 0, 1, \dots, [\frac{n}{2}]$ , the Euler-Lagrange equation for the first variational formula of the total  $2p$ -th mean curvature  $\mathcal{M}_{2p}(f)$  is given by:

$$L_{2p} := -(n-2p)H_{2p+1}^f + pW_{2p-1} + p\tilde{Q}_{2p-2} = 0.$$

Here  $H_{2p+1}^f, W_{2p-1}$  are the same with those in Theorem 3.2, and

$$\tilde{Q}_{2p-2} = \sum_{i,A} \langle Q_{2p-2}^i, \nabla_{e_i} e_A \rangle e_A - \sum_\alpha \operatorname{div} \left( \sum_i q_{2p-2}^{i,\alpha} e_i \right) e_\alpha,$$

where  $Q_{2p-2}^i = \sum_\alpha q_{2p-2}^{i,\alpha} e_\alpha$  is defined in Theorem 3.2 and denote  $\tilde{Q}_{-2} = 0$ . Henceforth, we call  $M$  relatively  $2p$ -minimal if  $L_{2p} = 0$ .

*Proof.* It suffices to treat with the term involving covariant derivative of  $\nu$  in (3.10). Recall that  $\nu = \sum_A a_A e_A$  and  $Q_{2p-2}^i = \sum_\alpha q_{2p-2}^{i,\alpha} e_\alpha$ . Then

$$\begin{aligned} \sum_i \langle Q_{2p-2}^i, \nabla_{e_i} \nu \rangle &= \sum_i \left\langle Q_{2p-2}^i, \sum_A e_i(a_A) e_A + \sum_A a_A \nabla_{e_i} e_A \right\rangle \\ &= \sum_{i,\alpha} q_{2p-2}^{i,\alpha} e_i(a_\alpha) + \sum_{i,A} a_A \langle Q_{2p-2}^i, \nabla_{e_i} e_A \rangle \\ &= \sum_\alpha \operatorname{div} \left( \sum_i a_\alpha q_{2p-2}^{i,\alpha} e_i \right) - \left\langle \sum_\alpha \operatorname{div} \left( \sum_i q_{2p-2}^{i,\alpha} e_i \right) e_\alpha, \nu \right\rangle + \left\langle \sum_{i,A} \langle Q_{2p-2}^i, \nabla_{e_i} e_A \rangle e_A, \nu \right\rangle \\ &= \sum_\alpha \operatorname{div} \left( \sum_i a_\alpha q_{2p-2}^{i,\alpha} e_i \right) + \langle \tilde{Q}_{2p-2}, \nu \rangle. \end{aligned}$$

Thus according to Stokes' theorem, one can easily find that

$$\left. \frac{d}{dt} \mathcal{M}_{2p}(f_t) \right|_{t=0} = \int_M \sum_\alpha \operatorname{div} \left( \sum_i a_\alpha q_{2p-2}^{i,\alpha} e_i \right) dV_M + \int_M \langle L_{2p}, \nu \rangle dV_M = \int_M \langle L_{2p}, \nu \rangle dV_M,$$

which completes the proof of the theorem.  $\square$

When  $N$  is a real space form  $\mathbb{R}^{n+m}(c)$  with constant sectional curvature  $c$ , one can find that

$$(3.11) \quad L_{2p} = -(n-2p)H_{2p+1}^f + 2cpH_{2p-1}^f,$$

which was proved by [13] firstly with different notations.



4. CLOSED COMPLEX SUBMANIFOLDS IN  $\mathbb{C}P^{n+m}$ 

In this section we prove that closed complex submanifolds in complex projective spaces are relatively  $2p$ -minimal, *i.e.*, critical for the functional  $\mathcal{M}_{2p}$  for all  $p$ .

Let  $N$  be the complex projective space  $\mathbb{C}P^{n+m}(c)$  with constant holomorphic sectional curvature  $c$ . Denote by  $J, \langle \cdot, \cdot \rangle$  the almost complex structure and Hermitian metric respectively. It is well known that the curvature tensor of  $N$  can be written as

$$\begin{aligned} R(X, Y, Z, T) = & \frac{c}{4} \left( \langle X, Z \rangle \langle Y, T \rangle - \langle Y, Z \rangle \langle X, T \rangle \right. \\ & \left. + \langle JX, Z \rangle \langle JY, T \rangle - \langle JY, Z \rangle \langle JX, T \rangle + 2 \langle JX, Y \rangle \langle JZ, T \rangle \right). \end{aligned}$$

Suppose  $M$  is a complex submanifold of complex dimension  $n$  in  $N$ . Around each point  $x$  in  $M$ , we can choose a local orthonormal frame  $\{e_1, \dots, e_{2n+2m}\}$  of  $TN$  such that  $e_2 = Je_1, \dots, e_{2n+2m} = Je_{2n+2m-1}$ , and  $e_1, \dots, e_{2n}$  are tangent to  $M$ . In this section, we still use  $i, j, k$  (resp.  $\alpha, \beta, \gamma$ ), *etc.* for the indices of tangent (resp. normal) vectors of  $M$ . In addition, for simplicity we will use the following notations

$$\bar{e}_i := e_{\bar{i}} := Je_i, \quad \bar{e}_\alpha := e_{\bar{\alpha}} := Je_\alpha.$$

Under this setting we can write the curvature tensor of  $N$  over  $M$  in a simpler form. For example,

$$(4.1) \quad R_{i\alpha jk} = 0, \quad R_{i\alpha j\beta} = \frac{c}{4} (\delta_j^i \delta_\beta^\alpha + \delta_j^{\bar{i}} \delta_\beta^{\bar{\alpha}}).$$

The following Lemmas will be useful in the proof of Theorem 4.3 later.

**Lemma 4.1.** *With the same notations as above, we get the following identity about the second fundamental form of  $M$ :*

$$(4.2) \quad \theta_{i\alpha}(e_j) = \theta_{i\bar{\alpha}}(\bar{e}_j) = -\theta_{\bar{i}\alpha}(\bar{e}_j).$$

*Proof.* Straightforward computation shows

$$\theta_{i\bar{\alpha}}(\bar{e}_j) = \theta_{j\bar{\alpha}}(e_i) = \langle \nabla_{e_i} \bar{e}_j, \bar{e}_\alpha \rangle = \langle J \nabla_{e_i} e_j, J e_\alpha \rangle = \langle \nabla_{e_i} e_j, e_\alpha \rangle = \theta_{i\alpha}(e_j).$$

Similarly,

$$\theta_{i\alpha}(\bar{e}_j) = \langle \nabla_{\bar{e}_j} \bar{e}_i, e_\alpha \rangle = -\langle \nabla_{\bar{e}_j} e_i, \bar{e}_\alpha \rangle = -\langle \nabla_{e_i} \bar{e}_j, \bar{e}_\alpha \rangle = -\langle \nabla_{e_i} e_j, e_\alpha \rangle = -\theta_{i\alpha}(e_j).$$

□

**Lemma 4.2.** *With the same notations as above, we get the following identity:*

$$\sum_s \Omega_{I_{2p}}(X_1, \dots, JX_s, \dots, X_{2p}) = 0,$$

where  $\Omega_{I_{2p}}$  is defined in (3.8),  $X_1, \dots, X_{2p}$  are  $2p$  vectors tangent to  $M$ .

*Proof.* Obviously  $M$  is also a Kähler manifold. Thus the formula  $\Omega_{ij}(JX_1, JX_2) = \Omega_{ij}(X_1, X_2)$  holds. We prove this Lemma by induction. For  $p = 1$ , it is not difficult to see that

$$\Omega_{i_1 i_2}(JX_1, X_2) + \Omega_{i_1 i_2}(X_1, JX_2) = 0.$$

Suppose the identity holds for  $p - 1$ , then for  $p$ ,

$$\sum_s \Omega_{I_{2p}}(X_1, \dots, JX_s, \dots, X_{2p})$$

$$\begin{aligned}
&= \sum_{t < s} (-1)^{s+t-1} \Omega_{I_{2p-2}}(X_1, \dots, \hat{X}_t, \dots, \hat{X}_s, \dots, X_{2p}) \Omega_{i_{2p-1}i_{2p}}(X_t, JX_s) \\
&\quad + \sum_{t > s} (-1)^{s+t-1} \Omega_{I_{2p-2}}(X_1, \dots, \hat{X}_s, \dots, \hat{X}_t, \dots, X_{2p}) \Omega_{i_{2p-1}i_{2p}}(JX_s, X_t) \\
&\quad + \sum_s \sum_{t_1, t_2 \neq s, t_1 < t_2} (-1)^{t_1+t_2-1} \Omega_{I_{2p-2}}(X_1, \dots, \hat{X}_{t_1}, \dots, \hat{X}_{t_2}, \dots, JX_s, \dots, X_{2p}) \Omega_{i_{2p-1}i_{2p}}(X_{t_1}, X_{t_2}) \\
&= \sum_{t < s} (-1)^{s+t-1} \Omega_{I_{2p-2}}(X_1, \dots, \hat{X}_t, \dots, \hat{X}_s, \dots, X_{2p}) \left( \Omega_{i_{2p-1}i_{2p}}(X_t, JX_s) + \Omega_{i_{2p-1}i_{2p}}(JX_t, X_s) \right) \\
&\quad + \sum_{t_1 < t_2} \sum_{s \neq t_1, t_2} (-1)^{t_1+t_2-1} \Omega_{I_{2p-2}}(X_1, \dots, \hat{X}_{t_1}, \dots, \hat{X}_{t_2}, \dots, JX_s, \dots, X_{2p}) \Omega_{i_{2p-1}i_{2p}}(X_{t_1}, X_{t_2}) \\
&= \sum_{t_1 < t_2} \sum_{s \neq t_1, t_2} (-1)^{t_1+t_2-1} \Omega_{I_{2p-2}}(X_1, \dots, \hat{X}_{t_1}, \dots, \hat{X}_{t_2}, \dots, JX_s, \dots, X_{2p}) \Omega_{i_{2p-1}i_{2p}}(X_{t_1}, X_{t_2}).
\end{aligned}$$

By assumption, the sum  $\sum_{s \neq t_1, t_2} \Omega_{I_{2p-2}}(X_1, \dots, \hat{X}_{t_1}, \dots, \hat{X}_{t_2}, \dots, JX_s, \dots, X_{2p})$  equals zero for all  $t_1, t_2$ . In conclusion, the proof is complete.  $\square$

**Theorem 4.3.** *Let  $M$  be a closed complex submanifold of complex dimension  $n$  in  $\mathbb{C}P^{n+m}$ , then*

$$L_{2p} = -(2n - 2p)H_{2p+1}^f + \frac{cp(2n - 2p)}{2(2n - 2p + 1)}H_{2p-1}^f = 0,$$

i.e.,  $M$  is relatively  $2p$ -minimal for  $p = 0, 1, \dots, n$ .

*Proof.* Clearly  $\tilde{Q}_{2p-2} = 0$  since now  $R_{i\alpha jk} = 0$  by (4.1). Therefore to calculate  $L_{2p}$  in Theorem 3.4, it suffices to compute  $W_{2p-1}$ . Combining the definition of  $H_{2p-1}^f$  and Lemma 4.1, we compute it as follows:

$$\begin{aligned}
W_{2p-1} &= \frac{2(2n - 2p)!}{(2p - 1)!(2n)!} \sum_{\alpha, \beta} \sum_{I_{2p-1}, i} \sum_{J_{2p-1}, j} \delta_{j_1, \dots, j_{2p-1}, j}^{i_1, \dots, i_{2p-1}, i} R_{i\beta j\alpha} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\beta}(e_{j_1}, \dots, e_{j_{2p-1}}) e_\alpha \\
&= \frac{c(2n - 2p + 1)!}{2(2p - 1)!(2n)!} \sum_{\alpha} \sum_{I_{2p-1}} \sum_{J_{2p-1}} \delta_{j_1, \dots, j_{2p-1}}^{i_1, \dots, i_{2p-1}} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\alpha}(e_{j_1}, \dots, e_{j_{2p-1}}) e_\alpha \\
&\quad + \frac{c(2n - 2p)!}{2(2p - 1)!(2n)!} \sum_{\alpha} \sum_{I_{2p-1}, i} \sum_{J_{2p-1}} \delta_{j_1, \dots, j_{2p-1}, i}^{i_1, \dots, i_{2p-1}, i} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\bar{\alpha}}(e_{j_1}, \dots, e_{j_{2p-1}}) e_\alpha \\
&= \frac{c}{2} H_{2p-1}^f - \frac{c(2n - 2p)!}{2(2n)!} \sum_{\alpha} \sum_{I_{2p-1}} \sum_{s=1}^{2p-1} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\bar{\alpha}}(e_{i_1}, \dots, \bar{e}_{i_s}, \dots, e_{i_{2p-1}}) e_\alpha \\
&\triangleq \frac{c}{2} H_{2p-1}^f - \frac{c(2n - 2p)!}{2(2n)!} \sum_{\alpha} \sum_{I_{2p-1}} \sum_{s=1}^{2p-1} (-1)^{s-1} \Omega_{I_{2p-2}}(e_{i_1}, \dots, \hat{e}_{i_s}, \dots, e_{i_{2p-1}}) \theta_{i_{2p-1}\bar{\alpha}}(\bar{e}_{i_s}) e_\alpha \\
&= \frac{c}{2} H_{2p-1}^f - \frac{c(2n - 2p)!}{2(2n)!} \sum_{\alpha} \sum_{I_{2p-1}} \sum_{s=1}^{2p-1} (-1)^{s-1} \Omega_{I_{2p-2}}(e_{i_1}, \dots, \hat{e}_{i_s}, \dots, e_{i_{2p-1}}) \theta_{i_{2p-1}\alpha}(e_{i_s}) e_\alpha \\
&= \frac{c}{2} H_{2p-1}^f - \frac{c(2n - 2p)!}{2(2n)!} \sum_{\alpha} \sum_{I_{2p-1}} \Omega_{I_{2p-2}} \wedge \theta_{i_{2p-1}\alpha}(e_{i_1}, \dots, e_{i_{2p-1}}) e_\alpha \\
&= \frac{c(2n - 2p)}{2(2n - 2p + 1)} H_{2p-1}^f,
\end{aligned}$$

where “ $\triangleq$ ” is deduced by Lemma 4.2. Therefore, we obtain

$$L_{2p} = -(2n - 2p)H_{2p+1}^f + \frac{cp(2n - 2p)}{2(2n - 2p + 1)}H_{2p-1}^f.$$

Meanwhile, a direct calculation shows that  $H_{2p+1}^f$  of  $M$  vanishes for each  $p$ . In fact, combining the fact that  $\Omega_{\bar{i}\bar{j}}(\bar{e}_k, \bar{e}_l) = \Omega_{ij}(e_k, e_l)$  with Lemma 4.1, we get

$$\begin{aligned} H_{2p+1}^f &= \frac{(n - 2p - 1)!}{n!} \sum_{\alpha} \sum_{I_{2p+1}} \Omega_{I_{2p}} \wedge \theta_{i_{2p+1}\alpha}(e_{i_1}, \dots, e_{i_{2p+1}}) e_{\alpha} \\ &= \frac{(n - 2p - 1)!}{n!} \sum_{\alpha} \sum_{I_{2p+1}} \Omega_{\bar{i}_1 \bar{i}_2} \wedge \dots \wedge \Omega_{\bar{i}_{2p-1} \bar{i}_{2p}} \wedge \theta_{\bar{i}_{2p+1}\alpha}(\bar{e}_{i_1}, \dots, \bar{e}_{i_{2p+1}}) e_{\alpha} \\ &= \frac{(n - 2p - 1)!}{n!} \sum_{\alpha, s} \sum_{I_{2p+1}} (-1)^{s-1} \Omega_{\bar{i}_1 \bar{i}_2} \wedge \dots \wedge \Omega_{\bar{i}_{2p-1} \bar{i}_{2p}}(\bar{e}_{i_1}, \dots, \hat{e}_{i_s}, \dots, \bar{e}_{i_{2p+1}}) \theta_{\bar{i}_{2p+1}\alpha}(\bar{e}_{i_s}) e_{\alpha} \\ &= -\frac{(n - 2p - 1)!}{n!} \sum_{\alpha, s} \sum_{I_{2p+1}} (-1)^{s-1} \Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{2p-1} i_{2p}}(e_{i_1}, \dots, \hat{e}_{i_s}, \dots, e_{i_{2p+1}}) \theta_{i_{2p+1}\alpha}(e_{i_s}) e_{\alpha} \\ &= -\frac{(n - 2p - 1)!}{n!} \sum_{\alpha} \sum_{I_{2p+1}} \Omega_{I_{2p}} \wedge \theta_{i_{2p+1}\alpha}(e_{i_1}, \dots, e_{i_{2p+1}}) e_{\alpha} \\ &= -H_{2p+1}^f = 0. \end{aligned}$$

This completes the proof of the theorem.  $\square$

## 5. RELATIVELY $2p$ -MINIMAL AND AUSTERE SUBMANIFOLDS

In this section, we discuss the relations between relatively  $2p$ -minimal submanifolds and austere submanifolds in real space forms, as well as a special variational problem.

Let  $f : M^n \rightarrow \mathbb{R}^{n+m}(c)$  be an isometric immersion in a real space form of constant sectional curvature  $c$ . Recall that the volume of any tubular hypersurface  $M^f(r)$  with radius  $r$  ( $0 < r < \varepsilon$ ) of  $M^n$  in  $\mathbb{R}^{n+m}(c)$  is given by the well known Weyl-Gray tube formula (cf. [7])

$$(5.1) \quad V(M^f(r)) = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \frac{C_{m+2p-1}}{2^{2p} \pi^p p!} \binom{n}{2p} (2p)! \mathcal{M}_{2p}(f) (\cos(r\sqrt{c}))^{n-2p} \left( \frac{\sin(r\sqrt{c})}{\sqrt{c}} \right)^{m+2p-1},$$

where  $C_{m+2p-1}$  is the volume of  $S^{m+2p-1}(1)$ , the  $\sin$ ,  $\cos$  functions are considered as complex functions, and  $\mathcal{M}_{2p}(f)$  is the total  $2p$ -th mean curvature of  $f$ . Put  $\mathcal{V}_r(f) := V(M^f(r))$ . Then  $\{\mathcal{V}_r \mid 0 < r < \varepsilon\}$  forms a one-parameter family of functionals over isometric submanifolds in  $\mathbb{R}^{n+m}(c)$ . We call  $M$  a *tubular minimal* submanifold of  $\mathbb{R}^{n+m}(c)$  if it is a critical point of  $\mathcal{V}_r$  for all  $0 < r < \varepsilon$ . Observing the Weyl-Gray tube formula (5.1), we find that  $M$  is a critical point of  $\mathcal{V}_r$  for all  $0 < r < \varepsilon$  if and only if it is a critical point of  $\mathcal{M}_{2p}$  for all  $p = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$ , or equivalently, it is  $2p$ -minimal for all  $p = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$ . Combining these with the Euler-Lagrange equation (3.11) and the second identity in (2.3), we deduce the following

**Proposition 5.1.** *Let  $f : M^n \rightarrow \mathbb{R}^{n+m}(c)$  be an isometric immersion in a real space form of constant sectional curvature  $c$ . Then the following are equivalent:*

- (i)  $M$  is tubular minimal;
- (ii)  $M$  is relatively  $2p$ -minimal, i.e.,  $L_{2p} = -(n - 2p)H_{2p+1}^f + 2cpH_{2p-1}^f = 0$  for all  $p = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$ ;

- (iii)  $H_{2p+1}^f = 0$  for all  $p = 0, 1, \dots, [\frac{n}{2}]$ ;
- (iv)  $M$  is  $2p$ -minimal, i.e.,  $H_{2p+1}^M = 0$  for all  $p = 0, 1, \dots, [\frac{n}{2}]$ .

Recall that a submanifold of a Riemannian manifold is called *austere* by Harvey and Lawson [10] if its principle curvatures in any normal direction occur in oppositely signed pairs. They showed, among other fundamental results on calibrated geometry, austere submanifolds of Euclidean space are exactly those whose co-normal bundle is special lagrangian and hence absolutely minimizing. Except for the case of surfaces, austerity is much stronger than minimality. Many examples and (partially) classifications of austere submanifolds of Euclidean space have been established by several authors, such as [1], [5], [11], *etc.* For minimal 3-folds in different space forms, [4] gives a local classification of the submanifolds for which the equality in the DDVV inequality (also called the normal scalar curvature inequality which was proved independently and differently by [16] and [8]) holds everywhere and hence austere. Note that by the pointwise equality condition for the DDVV inequality given by [8] (also discussed in [16]), minimality together with this DDVV equality is sufficient for austerity. It is worthy to mention that the classification problem of submanifolds attaining the DDVV equality everywhere still remains a rather interesting open problem (see [9] for a brief introduction and [16], [6] for some advances). As far as we compare austerity with tubular minimality, we derive the following

**Proposition 5.2.** *Let  $M^n$  be an  $n$ -dimensional austere submanifold of the real space form  $\mathbb{R}^{n+m}(c)$ . Then  $M$  is tubular minimal. Moreover, each  $2p$ -th mean curvature satisfies  $(-1)^p K_{2p}^f \geq 0$ .*

*Proof.* By the definition of austerity, we see that each odd order elementary symmetric polynomial  $M_{2p+1}(\xi)$  of the shape operator  $S_\xi$  with respect to any unit normal vector  $\xi$  of  $M$  vanishes. Recalling that in [7] it is proved that

$$H_{2p+1}^f = \frac{2^{2p}\pi^p p!(m+2p)}{C_{m+2p-1}(2p+1)!} \int_{S^{m-1}(1)} \xi M_{2p+1}(\xi) d\xi,$$

we get  $H_{2p+1}^f = 0$  for all  $p = 0, 1, \dots, [\frac{n}{2}]$ , and hence by Proposition 5.1,  $M$  is tubular minimal. Moreover, austerity implies that the  $2p$ -th elementary symmetric polynomial  $M_{2p}(\xi)$  of the shape operator  $S_\xi$  has the sign of  $(-1)^p$ , which then shows  $(-1)^p K_{2p}^f \geq 0$  by the following integral formula (cf. [7]):

$$K_{2p}^f = \frac{2^{2p}\pi^p p!}{C_{m+2p-1}(2p)!} \int_{S^{m-1}} M_{2p}(\xi) d\xi.$$

The proof is now complete. □

**Acknowledgements 1.** It is our great pleasure to thank Professor Zizhou Tang for his guidance and support. Many thanks also to Professors Haizhong Li, Jiagui Peng and Changping Wang for their helpful discussions and useful suggestions during the preparation of this paper.

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